Lec 2: 08/21/2013

Friedmann-Robertson-Walker Universe:

The theoretical framework for describing the evolution of our universe is Einstein's general theory of relativity. In this theory the geometry of the spacetime is related to the energy distribution according to Einstein's equation:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$ (c: speed of light, G: Newton's gravitational constant)

$G_{\mu\nu}$ is called Einstein tensor that depends on the spacetime metric $g_{\mu\nu}$ as follows:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$R_{\mu\nu} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{k}_{\mu\lambda} \Gamma^{\lambda}_{\nu k} - \Gamma^{k}_{\mu\nu} \Gamma^{\lambda}_{\lambda k}$$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left[\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right]$$

$R_{\mu\nu}$ and R ($R \equiv g^{\mu\nu} R_{\mu\nu}$) are called Ricci tensor and Ricci scalar respectively. $\Gamma^{\lambda}_{\mu\nu}$ is the affine connection (but not a tensor). Recall that from the metric we can find the distance between two points in the spacetime x^μ, $x^\mu + dx^\mu$:

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

The spacetime corresponding to our universe is 4-dimensional, hence $\mu = 0, 1, 2, 3$. Metric $g_{\mu\nu}$ is therefore given by a 4x4 matrix. Note that $g^{\mu\nu}$ is given by the inverse of this matrix $[g^{\mu\nu}] \equiv [g_{\mu\nu}]^{-1}$.

In general, finding the exact solutions of Einstein's equation are very difficult as it is a highly nonlinear equation. The situation gets substantially easier by using the observational evidence that our universe is highly isotropic and homogeneous at sufficiently large scales. We can consider a spacelike hypersurface in the spacetime that is homogeneous and isotropic, and ask how it evolves as a function of time.

There are three distinct solutions for a homogeneous and isotropic 3-dimensional space:

$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$ R^3, no curvature (flat)

$ds^2 = dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta\, d\varphi^2)$ S^3, positively curved (closed)

$ds^2 = dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta\, d\varphi^2)$ H^3, negatively curved (open)

S^3 and H^3 are 3-dimensional analogues of a two-sphere and a two-hyperbola respectively.

Starting with a homogeneous and isotropic hypersurface that contains a homogeneous and isotropic distribution of energy (given by the energy-momentum tensor), it will always remain homogeneous and isotropic on physical grounds. Therefore, the distance between two points on such a hypersurface can have a time dependence that shows up as an overall scale factor $a(t)$. Including the time coordinate, the metric for a homogeneous and isotropic universe becomes;

$$ds^2 = -c^2dt^2 + a^2(t)\begin{cases} dr^2 + r^2(\sin^2\theta\, d\varphi^2 + d\theta^2) & \text{flat} \\ dr^2 + \sin^2 r\,(\sin^2\theta\, d\varphi^2 + d\theta^2) & \text{closed} \\ dr^2 + \sinh^2 r\,(\sin^2\theta\, d\varphi^2 + d\theta^2) & \text{open} \end{cases}$$

The task here is to find time evolution of $a(t)$. In order to find this, one needs to solve Einstein's equation. This requires knowledge of the energy-momentum tensor. For a homogeneous and isotropic universe, it has a simple form:

$$T_{\nu\nu} = g_{\nu\nu}\, P + (\rho + P)\, v_\nu v_\nu \qquad v_\nu \equiv g_{\nu\rho} \frac{dx^\rho}{ds}$$

Here P and ρ are the pressure and energy density of the content of the universe, which is a "perfect fluid" due to homogeneity and istropicity. $\frac{dx^\rho}{ds}$ represents the motion of an element of the fluid. In a homogeneous and isotropic universe, there can be no bulk motion of the fluid, which implies only diagonal entries for $T_{\nu\nu}$.

Taking these into consideration, Einstein's equation is reduced to two equation called Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3c^2} - \frac{Kc^2}{a^2} \qquad (\mathrm{I}) \qquad K = \begin{cases} +1 & \text{closed} \\ 0 & \text{flat} \\ -1 & \text{open} \end{cases}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) \qquad (\mathrm{II})$$

The Hubble expansion rate is defined as:

$$H \equiv \frac{\dot{a}}{a}.$$

Before finding explicit solutions of the Friedmann equations, lets discuss two general observations that can be made without solving the equations:

1) The first equation (I) involves $\left(\frac{\dot{a}}{a}\right)^2$. Therefore the sign of $\dot{a}$ is not determined from the Friedmann equation. However, it is seen that a closed universe ($k=+1$) that expands ($\dot{a}>0$), will stop expanding for known distributions of energy ($\rho_{(t)} \propto a^{-3}_{(t)}$ for non-relativistic matter, $\rho_{(t)} \propto a^{-4}_{(t)}$ for relativistic particles). After that, one has $\dot{a}<0$, which implies a contracting universe. As we will see later, the situation will be different if the universe is filled with a fluid for which $\rho_{(t)} = $ const, at all times.

For a flat ($k=0$) or open ($k=-1$) universe, the expansion will never stop as $t \to \infty$.

(2) From the second Friedmann equation (II) we see that $\ddot{a} < 0$ as long as $\rho + 3p < 0$. For known distribution of energy this is indeed the case ($p = 0$ for non-relativistic particles, $p = \frac{1}{3}\rho$ for relativistic particles). This implies that in an expanding universe expansion will decelerate for such perfect fluids.

Accelerated expansion requires negative pressure $p < -\frac{1}{3}\rho$. The simplest possibility is the so-called cosmological constant for which $p_{(t)} = -\rho_{(t)} = \text{Const.}$, which we will discuss later. Since there is observational evidence that our universe has lately entered a phase of accelerated expansion, this implies a peculiar component has dominated the universe (called "dark energy") for which $p < -\frac{1}{3}\rho$.

An important point to note is that Friedman equations on page 12 are written for an FRW universe filled with one type of fluid. In general, various components of energy can exist together (matter, radiation, dark energy, etc).

The general form of Friedmann equations is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \sum_i \frac{8\pi G \rho_i}{3c^2} - \frac{kc^2}{a^2} \quad (\text{III})$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \sum_i (\rho_i + 3 P_i) \quad (\text{IV})$$

Here ρ_i, P_i denote the energy density and pressure of the i-th component of the energy content.

However, it usually happens that at a given time, one of the components is dominant over the others. For example, at very early times, the universe was dominated by relativistic particles ($P = \frac{1}{3}\rho$), and then by matter ($P = 0$) for several billion years. Lately (for the past ~4 billion years), it has been dominated by dark energy ($P \simeq -\rho$). Once a certain component is dominant, the equations (III, IV) will be reduced to equations (I, II) with ρ, P being those of the dominating component. As a result, the Friedmann equations (I, II) suffice to find $a(t)$ for most parts of the evolution of the universe. However, to describe transition from one component to another one, we must use equations (III, IV).